Reply to 'Comment on "Equilibrium crystal shape of the Potts model at the first-order transition point"'

2002 J. Phys. A: Math. Gen. 35 7553

(http://iopscience.iop.org/0305-4470/35/34/402)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 7553–7557 PII: S0305-4470(02)36224-3

REPLY

Reply to 'Comment on "Equilibrium crystal shape of the Potts model at the first-order transition point"'

Masafumi Fujimoto

Department of Physics, Nara Medical University, Kashihara, Nara 634-8521, Japan

Received 19 June 2002
Published 15 August 2002
Online at stacks.iop.org/JPhysA/35/7553

Abstract

The eight-vertex model is defined on the square lattice rotated through an arbitrary angle with respect to the coordinate axes. We re-examine the analysis of the anisotropic correlation length in a previous paper (Fujimoto M 1996 *Physica* A **233** 485–502). We point out that the asymptotic form of the correlation function is expressed by the use of differential forms on a Riemann surface of genus 1. Combined with the symmetry of the square lattice, this fact explains that the anisotropic correlation length is represented in terms of simple algebraic curves. The argument is applicable to a wide class of lattice models (including unsolvable ones).

PACS numbers: 05.50.+q, 02.10.−v, 02.20.−a, 02.40.−k, 61.50.Ah

In a previous paper [1] we considered the Q-state Potts model on the square lattice. It was shown that the anisotropic correlation length (ACL) is related by duality to the anisotropic interfacial tension (AIT). For $Q > 4$ the ACL was calculated at the first-order transition (or self-dual) point. From the calculated ACL the equilibrium crystal shape (ECS) was derived via the duality relation and the Wulff construction. The ECS was represented as

$$\alpha^2\beta^2 + 1 + A_3(\alpha^2 + \beta^2) + A_4\alpha\beta = 0 \tag{1a}$$

with

$$\alpha = \exp[-\Lambda(X + Y)/k_B T] \qquad \beta = \exp[-\Lambda(X - Y)/k_B T] \tag{1b}$$

for definitions of A_3 and A_4, see section 3.2 of [1]. The algebraic curve is a quite general one which appears as the ECS of a wide class of square-lattice models including the eight-vertex model [2].

In his comment [3] Rutkevich showed that the one-particle dispersion relations in the six-vertex model, the Ising model, and the Gaussian model are the same; note that the self-dual Potts model is equivalent to the six-vertex model. He argued as follows: one can say that the algebraic curve (1) reflects the universality in the one-particle dispersion relation; it is likely that the origin of the universality lies in the symmetry of the square lattice. An extension of the argument was suggested: the ACL of the Ising model on the cubic lattice could be identical with that of the Gaussian model on the cubic lattice.

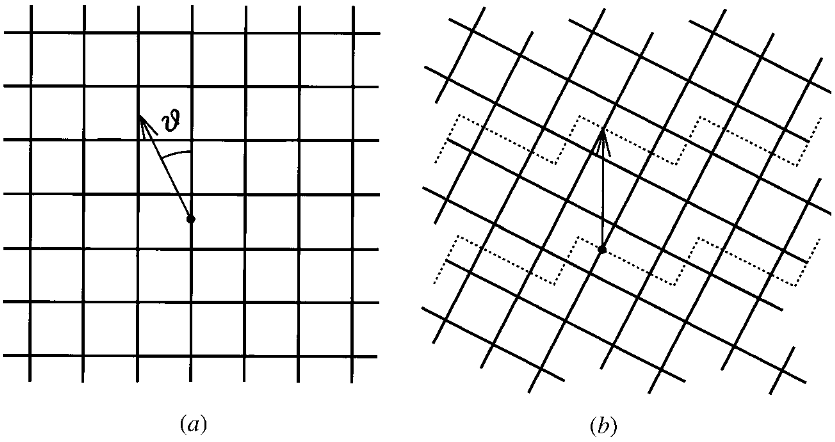

Figure 1. (a) Method of active rotation. Keeping the square lattice fixed, we investigate along the direction designated by θ with the help of the row-to-row transfer matrix and the shift operator. (b) Method of passive rotation. The square lattice is rotated through θ. The rotated system is investigated by the use of the transfer matrix $\tilde{\mathbf{V}}$ acting on zigzag walls.

In the beginning of this reply we point out that (1) is not the only universal curve. For example, we calculated the ACL of the square-lattice eight-vertex model in [4]. It was found that for a given x $(0 < x < 1)$ there are two cases with respect to a parameter q. In the case $0 < q < x^3$ the ACL is written by use of the algebraic curve (1); thus the argument in [3] is applicable. For $x^3 < q < x^4$ the ACL is not related to (1), but to the algebraic curve

$$\alpha^2\beta^2 + 1 + \bar{A}_2(\alpha\beta + 1)(\alpha + \beta) + \bar{A}_3(\alpha^2 + \beta^2) + \bar{A}_4\alpha\beta = 0. \tag{2}$$

The algebraic curve (2) comes from the dispersion relation of bound states of two free particles; the bound states give the next-next-largest eigenvalues of the transfer matrix. We note that the ACL for $x^3 < q < x^4$ is the same as that of the Ising model on the Union Jack (or 4-8) lattice. Some authors derived algebraic curves for lattice models possessing six-fold rotational symmetry; see [5, 6] and references therein. These algebraic curves are also universal.

How do symmetries of lattice models select the algebraic curves? This is the problem we must consider. In this reply we investigate the eight-vertex model defined on the square lattice rotated through an arbitrary angle with respect to the coordinate axes [7, 8] (figure 1(b)). The analysis of the ACL in [4] is re-examined. We find that the asymptotic form of the correlation function is expressed in terms of differential forms on a Riemann surface of genus 1. Combined with the symmetry of the square lattice, this fact explains that the ACL is represented by the use of the algebraic curve (1) or (2).

In [4] we denoted by θ the direction along which the correlation length was calculated; $\tan\theta = -m/l$. Since the system possesses inversion symmetry, we restricted ourselves to $-\pi/2 < \theta < \pi/2$, $l > 0$, $-\infty < m < \infty$ without loss of generality. To find the correlation length along the direction θ, we consider the square lattice rotated through θ. The rotated system is investigated with the help of an inhomogeneous transfer matrix defined by

$$\begin{aligned}[\mathbf{V}_{\mathrm{IH}}(u)]_{\alpha,\beta} = \sum_{\mu}\prod_{i=0}^{k-1}\Bigg[&\prod_{j=i(l+|m|)}^{i(l+|m|)+l-1} W(\mu_{j+1}, \alpha_{j+1}|\beta_{j+1}, \mu_{j+2}|u) \\ &\times \prod_{n=i(l+|m|)+l}^{(i+1)(l+|m|)-1} W(\mu_{n+1}, \alpha_{n+1}|\beta_{n+1}, \mu_{n+2}|u - v \mp \lambda)\Bigg]\end{aligned} \tag{3}$$

where $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_{k(l+|m|)}\}$ and $\beta = \{\beta_1, \beta_2, \ldots, \beta_{k(l+|m|)}\}$ are the arrow spins on two successive rows of vertical edges, and $\mu = \{\mu_1, \mu_2, \ldots, \mu_{k(l+|m|)}\}$ the arrow spins on a row intervening between α and β; for the definition of W, see (2.1) of [4]. In the second line the upper sign (respectively lower sign) corresponds to the case $m > 0$ (respectively $m < 0$).

It should be noted that different kinds of boundary conditions were mixed in [3]; for example, equations (3) of [3] relate the row-to-row transfer matrix T_2 and the shift operator T_1 of the Potts model with periodic boundary conditions to the two shift operators T_x, T_y of the six-vertex model with screw-shaped boundary conditions; see [9]. The mixture of boundary conditions makes it difficult to note some important points. In this reply we impose periodic boundary conditions in both directions: the same boundary conditions as we assumed in [4].

Using $\mathbf{V}_{\mathrm{IH}}(u)$, we construct two operators $\bar{\mathbf{V}}$ and $\bar{\mathbf{T}}$ as

$$\bar{\mathbf{V}} = \mathbf{V}^{l}_{\mathrm{IH}}(v)\mathbf{V}^{|m|}_{\mathrm{IH}}(\lambda) \qquad \bar{\mathbf{T}} = \begin{cases} \mathbf{V}^{m}_{\mathrm{IH}}(v)\mathbf{V}^{l}_{\mathrm{IH}}(-\lambda) & \text{for} \quad m > 0 \\ \mathbf{V}^{-m}_{\mathrm{IH}}(v-2\lambda)\mathbf{V}^{l}_{\mathrm{IH}}(-\lambda) & \text{for} \quad m < 0 \end{cases}. \tag{4}$$

The operator $\bar{\mathbf{V}}$ is equivalent to a transfer matrix acting on zigzag walls in the rotated system (figure 1(*b*)), and $\bar{\mathbf{T}}$ a shift operator [5–8]. The transfer matrix $\bar{\mathbf{V}}$ reduces to the row-to-row transfer matrix in the case $m = 0$, and to the diagonal-to-diagonal transfer matrix in the case $m = \pm l$. We can find the correlation length along the direction θ from the eigenvalues of $\bar{\mathbf{V}}$.

For the eigenvalues of $\mathbf{V}_{\mathrm{IH}}(u)$ the limiting function $L_{\mathrm{IH}}(u)$ is defined by (2.13) of [4]; we take the $k \to \infty$ limit with l and m fixed to be constants (instead of the $M \to \infty$ limit). The limiting function $L_{\mathrm{IH}}(u)$ is doubly periodic:

$$L_{\mathrm{IH}}(u + 4\mathrm{i}I) = L_{\mathrm{IH}}(u + 4\lambda) = L_{\mathrm{IH}}(u). \tag{5}$$

It follows that $L_{\mathrm{IH}}(u)$ satisfies the inversion relation

$$L_{\mathrm{IH}}(u + 2\lambda) = L^{-1}_{\mathrm{IH}}(u). \tag{6}$$

Assuming some analytic properties, and using (5) and (6), we can calculate explicit forms of $L_{\mathrm{IH}}(u)$ [10]. For some largest eigenvalues $L_{\mathrm{IH}}(u)$ is given by (2.14)–(2.18) of [4] with v replaced by u; Θ_1, Θ_2 and Θ are regarded as complex numbers.

The asymptotic form of the correlation function is obtained as (3.8) or (3.16) of [4]. Note that the rotation of the square lattice deforms the integration paths along the imaginary axis in [4] into those determined by the condition that the eigenvalues of the shift operator $\bar{\mathbf{T}}$ are unimodular:

$$\left|L^{m}_{\mathrm{IH}}(v)L^{l}_{\mathrm{IH}}(-\lambda)\right| = 1. \tag{7}$$

We cannot directly relate $\bar{\mathbf{V}}$ to the row-to-row transfer matrix in [4]. After the eigenvalues are summed up, however, results by $\bar{\mathbf{V}}$ are equivalent to those of [4]. This equivalence is derived with the help of the analyticity of the integrands in (3.8) and (3.16) of [4]. It follows that (i) analyticity of $L_{\mathrm{IH}}(u)$ (or $L(v)$ in [4]) is needed to ensure the equivalence between the two methods in figure 1.

Two further properties are pointed out. Considering the cases $\theta = \pm\pi/2$ shows that (ii) the inversion relation is connected with the inversion symmetry of the model. It is noted that (1) and (2) represent elliptic curves (i.e. they are algebraic curves of genus 1) [11]; see also section 4 of [6]. From (3.8) and (3.16) of [4] we find that (iii) the correlation function is written in terms of differential forms on a Riemann surface of genus 1.

The meaning of (iii) can be explained as follows: it is known that the two-dimensional lattice models are related to the two-dimensional Euclidean field theories in the critical limit and for distances much larger than the lattice spacing. For an Euclidean field the dispersion relation is written as $p_x^2 + p_y^2 + m^2 = 0$ with a suitable mass term m. The two-point correlation function is expressed in terms of a differential form on the rational curve. The differential form has a periodic structure describing the rotational symmetry. For lattice models two kinds of periods appear: one is connected with two-, four- or six-fold rotational symmetry, and the other with the fact that eigenvalues of the transfer matrix are periodic functions of crystal momentum. The doubly periodic structure indicates the property (iii).

Using (i)–(iii), we can reproduce the results in [4]. The property (iii) shows that, choosing a suitable Riemann surface of genus 1, we can write the correlation function as

$$\langle\alpha_{00}\alpha_{lm}\rangle - \langle\alpha_{00}\rangle\langle\alpha_{lm}\rangle \sim \int_0^{2\omega_1} \mathrm{d}\,\Theta\, F^l(\Theta)G^m(\Theta) \tag{8}$$

where $\mathrm{d}\Theta$ is a holomorphic (or analytic) differential form on the Riemann surface; $F(\Theta)$ comes from the eigenvalues of the row-to-row transfer matrix and $G(\Theta)$ from the eigenvalues of the shift operator; $F(\Theta)$ and $G(\Theta)$ are doubly periodic:

$$F(\Theta+2\omega_1) = F(\Theta+2\omega_2) = F(\Theta) \qquad G(\Theta+2\omega_1) = G(\Theta+2\omega_2) = G(\Theta). \tag{9}$$

From (ii) we obtain the inversion relation $F(\Theta + \omega_2) = F^{-1}(\Theta)$. The property (i) indicates the analytic properties of $F(\Theta)$. It follows that $F(\Theta)$ must be of the form

$$F(\Theta) = \prod_{i=1}^{\nu} ik^{1/2}\mathrm{sn}(\Theta+\alpha_i). \tag{10}$$

The eight-vertex model possesses four-fold rotational symmetry in a special limit. It follows that $G(\Theta) = F(\Theta + v)$ with a parameter v. We redefine Θ and v so that the condition $G(0) = 1$ is satisfied.

The case $\nu = 2$ corresponds to the square-lattice eight-vertex model. From the fact that the correlation function is a real-valued function, it follows that the modular parameter $\tau = \omega_2/\omega_1$ must be pure imaginary. For parameters α_1 and α_2 we find two possibilities: $\alpha_1 - \alpha_2$ is a pure imaginary or a real number. The former case gives (3.16) of [4] and the algebraic curve (2). In the latter case, integrating over $\alpha_1 - \alpha_2$, we find (3.8) of [4] and the algebraic curve (1).

We expect (i)–(iii) to be quite general properties satisfied by a wide class of two-dimensional lattice models (including unsolvable ones). Choosing the case $\nu = 4$ in (10), and using the six-fold rotational symmetry, we can explain the analysis of the Kagomé-lattice eight-vertex model in [5] .

The arguments from (8) to (10) indicate a close connection between symmetries of lattice models and covering problems of Riemann surfaces [11, 12]. I hope this point will be clarified in further publications.

Lastly, the Ising model on the cubic lattice is mentioned. For the simple-cubic nearest-neighbour Ising model, Holzer and Wortis [13] investigated the step tension and the facet shape by a low-temperature expansion. The calculated facet shape is not connected with the dispersion relation in the Gaussian model on the cubic lattice. It is quite doubtful whether the ACL of the cubic-lattice Ising model could be the same as that of the Gaussian model on the cubic lattice.

References

[1] Fujimoto M 1997 *J. Phys. A: Math. Gen.* **30** 3779–93
[2] Fujimoto M 1992 *J. Stat. Phys.* **67** 123–54
[3] Rutkevich S B 2002 *J. Phys. A: Math. Gen.* **35** 7549–52
[4] Fujimoto M 1996 *Physica* A **233** 485–502
[5] Fujimoto M 1998 *J. Stat. Phys.* **90** 363–88
[6] Fujimoto M 2002 *J. Phys. A: Math. Gen.* **35** 1517–31

[7] Fujimoto M 1994 *J. Phys. A: Math. Gen.* **27** 5101–19
[8] Fujimoto M 1996 *J. Stat. Phys.* **86** 1519–39
[9] Mizuta H, Nagao T and Wadati M 1994 *J. Phys. Soc. Japan* **63** 3951–61
[10] Klümper A and Zittartz J 1988 *Z. Phys.* B **71** 495–507
[11] Namba M 1984 *Geometry of Projective Algebraic Curves* (New York: Dekker)
[12] Namba M 1984 *Branched Coverings and Algebraic Functions* (Harlow: Longman Scientific & Technical)
[13] Holzer M and Wortis M 1989 *Phys. Rev.* B **40** 11044–58